Quantum superintegrable systems for arbitrary spin

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Abstract

In [1] was considered the superintegrable system which describes the magnetic dipole with spin $\frac{1}{2}$ (neutron) in the field of linear current. Here we present its generalization for any spin which preserves superintegrability . The dynamical symmetry stays the same as it is for spin $\frac{1}{2}$.

1 Introduction

There exist few quantum systems where the degeneration of spectrum is bigger when it follows from geometrical symmetry of the problem. The famous examples of such systems are isotropic oscillator, Kepler problem, rotator and some other which have no physical interpretation. This supplementary degeneration of the spectrum arises due to dynamical symmetry (which includes trivial geometrical). In this way the geometrical symmetry $SO(3)$ extends to the group $SU(3)$ in the case of of isotropic oscillator and to the group $SO(4)$ in case of bound spectrum od Kepler problem. This kind of systems also called maximally super-integrable. Their characteristic property is that all finite trajectories are closed. 30 years ago we with Stroganov had found another example of the physical system which possesses supplementary degeneration of its spectrum due to existence of hidden symmetry. The system describe the magnetic dipole with spin $\frac{1}{2}$ (neutron) in the field of line current. The obvious, geometrical symmetry is the symmetry $SO(2)$ with respect to rotation around z-axis, the direction of current (the translation along z is trivially separated). Dynamical group in this case is $SO(3)$. Here we are speaking about the symmetry for the negative part of the spectrum. For scattering states this group changes as in the case of Kepler problem and becomes the other real form of complex $SO(3)$, namely $SO(2,1)$ (or $E(2)$ for $E = 0$).

The peculiarity of the system which we discovered is that it describes the particle with spin, what was not know before. The question which was raised

soon after is whether it possible to preserve dynamical symmetry for the particles with higher spins. The answer up to now was negative in spite of many attempts [2][3][4]. The failure of previous considerations was because people wanted to preserve the interaction of the spin particle with the external field which corresponded to intuitive picture. But the truth is that particle with higher spin may interact not only by its dipole magnetic moment. For example, the particle with spin 1 acquires the possibility to have apart from dipole also quadruple interaction, for spin $\frac{3}{2}$– octuple interaction et cetera [1]. Certainly, this modification of interaction does not describe any longer an elementary particle like neutron in magnetic field of the linear wire. In the same time the emerged system could be useful for the description of trapped ultra cold atoms in this field — the subject was intensely discussed in the literature [5][6]. Apparently atoms, being extended objects will manifest its structure in inhomogeneous magnetic field through interaction much more complicated when dipole interaction of neutron. As we shall see below, the requirement of maximal super-integrability fixes the form of interaction up to finite number of the parameters — for spin s the number of parameters is $2s+1$. Accidentally or not, but almost all known maximally super-integrable systems have a wide physical application. The system which we considered in [1] was discussed in [7] in connection with the trapping of ultra-cold neutrons. So it could happened that the interaction we found for higher spins also includes some important cases.

The Hamiltonian of the system, considered in [1] is given by

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} - \mu \mathbf{H}, \tag{1}$$

where μ is magnetic moment of the particle and $\mathbf{H}$ is magnetic field of linear current directed along the z-axis:

$$\mathbf{H} = CI(\frac{y}{r^2}, -\frac{x}{r^2}). \tag{2}$$

The constant coefficient C depends on the unit system, in practical system $C = 0.2$. Thus the final form of the Hamiltonian will be

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} - k\frac{s_x y - s_y x}{r^2}, \tag{3}$$

[1] As a matter of fact, the importance of other interaction for higher spins manifests itself also in case of Heisenberg magnetic, which is integrable only if the interaction between spins is modified.

where the coefficient k collected all constants. For spin $\frac{1}{2}$ the spin operator proportional to Pauli matrices. The Hamiltonian (1) is invariant with respect to rotations around z-axis generated by $J_z = L_z + s_z$. In addition to this geometrical integral the Hamiltonian (3) possesses two non-trivial :

$$\begin{aligned} A_x &= \frac{1}{2}(J_3 p_x + p_x J_3) + km\frac{s_x y - s_y x}{r^2} y \\ A_y &= \frac{1}{2}(J_3 p_y + p_y J_3) - km\frac{s_x y - s_y x}{r^2} x \end{aligned} \tag{4}$$

The integrals (4) together with Hamiltonian and J_z form the following algebra:

$$\begin{gathered} [J_z, A_x] = iA_y, \quad [J_z, A_y] = -iA_x, \quad [A_x, A_y] = -i\mathcal{H}J_z \\ [A_x, \mathcal{H}] = 0, \quad [A_y, \mathcal{H}] = 0. \end{gathered} \tag{5}$$

If we define now the operators

$$J_x = A_x(-\mathcal{H})^{-1/2}, \qquad J_y = A_y(-\mathcal{H})^{-1/2}, \tag{6}$$

then the following commutation relations of $SO3$ algebra hold true:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{7}$$

While we designed the operators J_i we had in mind the the discrete spectrum, for which energy is negative. For positive energy the algebra will be $SO(2,1)$, because some signs in (7) will change. The Casimir operator of the algebra (7) is expressed via Hamiltonian:

$$\mathbf{J^2} = J_1^2 + J_2^2 + J_3^2 = -\frac{1}{4} - \frac{mk^2}{2\mathcal{H}}, \tag{8}$$

therefore the Hamiltonian is given by

$$\mathcal{H} = -\frac{mk^2}{2}\frac{1}{\mathbf{J^2} + \frac{1}{4}}. \tag{9}$$

The representations of $SO(3)$ characterized by integer or half-integer spin. In our problem it is clear that the eigenvalues of J_3 could be only half-integer

due to addition of spin $\frac{1}{2}$ and integer orbital momentum, therefore only half-integer representations will be realized. So the eigenvalues of $\mathbf{J^2}$ will be $\frac{2n+1}{2}(\frac{2n+1}{2}+1)$, $n = 0, 1...$ and the spectrum of energy will be

$$E_n = -\frac{mk^2}{2}\frac{1}{(n+1)^2} \tag{10}$$

The supplementary degeneration in this case means that the spectrum does not depends on the eigenvalue of J_z.

The existence of additional integrals of motion in this case based completely on the properties of Pauli matrices which represent spin $\frac{1}{2}$ operators for and direct substitution instead of it, the matrices which represent any other spin immediately destroys the whole construction. The exit of this situation we will discuss in the next section.

2 High spins

Let us consider the quantum system which describes neutral particle with arbitrary spin s in the field of rectilinear electric current. The general form of the Hamiltonian in this case is given by the following equation (here we omitted the trivial part of kinetic motion along the line of the current):

$$\mathcal{H} = \frac{\mathbf{p^2}}{2m} + \frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}} \tag{11}$$

where $\mathbf{p}, \mathbf{x}$ are 2-dimensional vectors, $\mathbf{s}$ is spin operator $\mathbf{s} = (s_x, s_y, s_z)$. The matrix $M(\mathbf{s}, \mathbf{x})$ will be specified later. Now we shall impose on $M(\mathbf{s}, \mathbf{x})$ only one condition

$$[M(\mathbf{s}, \mathbf{x}), J_z] = 0, \tag{12}$$

where $J_z = L_z + s_z$. Now let us look for the additional integrals of motion in the following form:

$$A_i = \frac{1}{2}(p_i J_z + J_z p_i) + V_i(\mathbf{s}, \mathbf{x}), \tag{13}$$

where $V_i(\mathbf{s}, \mathbf{x})$ is an operator which should be defined by interaction $M(\mathbf{s}, \mathbf{x})$. The commutator of the first term of (13) with Hamiltonian (11) gives

$$[\mathcal{H}, \frac{1}{2}(p_i J_z + J_z p_i)] = \frac{i}{2}\{J_z, \partial_i \frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}}\}, \tag{14}$$

where $\{A, B\} = AB + BA$. This form of commutator suggests the following structure of the operator $V_i(\mathbf{s}, \mathbf{x})$:

$$V_i(\mathbf{s}, \mathbf{x}) = \epsilon_{ij} \frac{x_i}{\mathbf{x^2}} M(\mathbf{s}, \mathbf{x}) \tag{15}$$

where ϵ_{ij} -antisymmetric tensor. Indeed, commuting (15) with Hamiltonian we obtain:

$$\begin{aligned}[][\mathcal{H}, \epsilon_{ij} \frac{x_i}{\mathbf{x^2}} M(\mathbf{s}, \mathbf{x})] = -\frac{i}{2m}[- \left\{ L_z, \partial_i \frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}} \right\} \\ + \left\{ p_k, \epsilon_{ik} (\frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}} + x_j \partial_j \frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}}) \right\}],\end{aligned} \tag{16}$$

Now if we add and subtract s_z to L_z we can rewrite (16) in the following form:

$$\begin{aligned}[][\mathcal{H}, \epsilon_{ij} \frac{x_i}{\mathbf{x^2}} M(\mathbf{s}, \mathbf{x})] = -\frac{i}{2m}[- \{ J_z, \partial_i \frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}} \} + \partial_i \{ s_z, \frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}} \} \\ + \left\{ p_k, \epsilon_{ik} (\frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}} + x_j \partial_j \frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}}) \right\}].\end{aligned} \tag{17}$$

Imposing on matrix $M(\mathbf{s}, \mathbf{x})$ apart from (12) the conditions

$$\begin{aligned} s_z M(\mathbf{s}, \mathbf{x}) + M(\mathbf{s}, \mathbf{x}) s_z = 0, \\ (\frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}} + x_j \partial_j \frac{M(\mathbf{s}, \mathbf{x})}{\mathbf{x^2}}) = 0,\end{aligned} \tag{18}$$

we arrive at the commutativity of

$$A_i = \frac{1}{2}(p_i J_z + J_z p_i) - m \epsilon_{ij} \frac{x_i}{\mathbf{x^2}} M(\mathbf{s}, \mathbf{x}) \tag{19}$$

with Hamiltonian. Note, that the matrix $M(\mathbf{s}, \mathbf{x})$, which we had in the previous section for spin $\frac{1}{2}$ satisfies both conditions (18) and it was the reason why we achieved the commutativity of integrals (4) with Hamiltonian. Now it is possible to prove that the commutation relations for the components of A_i are

$$[A_i, A_j] = -i \epsilon_{ij} J_z 2m \mathcal{H}, \tag{20}$$

provided the same conditions (12) and (18) are satisfied.

Now let us take care of matrix $M(\mathbf{s},\mathbf{x})$. The second equation (18) is rather simple, it requires $M(\mathbf{s},\mathbf{x})$ to be a homogenous function of x_i of degree 1. So we can present $M(\mathbf{s},\mathbf{x})$ in the form

$$M(\mathbf{s},\mathbf{x}) = |\mathbf{x}|\mu(\mathbf{s},\mathbf{n}), \qquad \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}, \tag{21}$$

where the matrix $\mu(\mathbf{s},\mathbf{n})$ commutes with J_z and anticommutes with s_z. Let us consider these conditions in the basis $|s,k>$ of the unitary representation of spin s. This basis is defined by

$$\begin{gathered} s_z|s,k>= k|s,k>, \qquad \mathbf{s^2}|s,k>= s(s+1)|s,k> \\ s_+|s,k>= \sqrt{s(s+1)-k(k+1)}|s,k+1>, \\ s_-|s,k>= \sqrt{s(s+1)-k(k-1)}|s,k-1> \\ k = s, s-1, ..., -s. \end{gathered} \tag{22}$$

In this basis $\mu(\mathbf{s},\mathbf{n})$ has its matrix elements $\mu_{kk'}(\mathbf{n})$

$$\mu_{kk'}(\mathbf{n}) =< s,k|\mu(\mathbf{s},\mathbf{n})|s,k'> . \tag{23}$$

The first equation (18) implies the following:

$$(k+k')\mu_{kk'}(\mathbf{n}) = 0. \tag{24}$$

The solution of this equation is

$$\mu_{kk'}(\mathbf{n}) = \delta_{k,-k'}a_k(\mathbf{n}), \qquad a_k^*(\mathbf{n}) = a_{-k}(\mathbf{n}), \tag{25}$$

where the last condition guaranties that $\mu(\mathbf{s},\mathbf{n})$ will be hermitian. Now let us impose the condition (12) on matrix $\mu(\mathbf{s},\mathbf{n})$:

$$[J_z,\mu(\mathbf{s},\mathbf{n})] = 0 \Rightarrow [L_z, a_k(\mathbf{n})] + 2ka_k(\mathbf{n}) = 0 \tag{26}$$

This equation fixes the $\mathbf{n}$-dependence of $a_k(\mathbf{n})$:

$$a_k(\mathbf{n}) = \alpha_k e^{-2ik\varphi}, \qquad e^{i\varphi} = n_1 + in_2, \qquad \alpha_k^* = \alpha_{-k}. \tag{27}$$

So, the final expression for matrix $\mu_{kk'}(\mathbf{n})$

$$\mu_{kk'}(\mathbf{n}) = \delta_{k,-k'}\alpha_k e^{-2ik\varphi} \tag{28}$$

contains $2s+1$ real parameters which define the set of α_k. The matrix $\mu_{kk'}(\mathbf{n})$ could also be expressed in terms of operators $\mathbf{s}$:

$$\mu(\mathbf{s},\mathbf{n}) = \Big(\beta_s (s_+n_-)^{2s} + h.c.\Big) + \Big(\beta_{s-1}(s_z - s)(s_+n_-)^{2s-2}(s_z+s) + h.c\Big)$$
$$+ \Big(\beta_{s-2}(s_z-s)(s_z-s+1)(s_+n_-)^{2s-4}(s_z+s)(s_z+s-1) + h.c\Big) \ldots \quad (29)$$

The structure of this expression could be understood from the following explanations. First, note that due to condition (26), matrix $\mu(\mathbf{s},\mathbf{n})$ can depend only of the combinations of $(s_+n_-), (s_-n_+)$ and s_z. Second, matrix $\mu(\mathbf{s},\mathbf{n})$ in the representation (22) is anti-diagonal and in order to obtain operator which has non zero matrix elements in the upper and lower corners we need to take a liner combination of (s_+n_-) and its conjugated in maximal power—for spin s it is $2s$. In this way we obtain the first term of (29). The second term is obtained with the same strategy but here we need to eliminate the action of $(s_+n_-)^{2s-2}$ on the vectors $|-s,s>$ and $<s,s|$. This is the reason of appearing the fringe multipliers $(s_z - s)$ and $(s_z + s)$. The rest is just repetition of this procedure. The parameters β_k in (29) play the same role, as α_k in (27) but only $\beta_s = \alpha_s$, the other are different because of additional multipliers, depending on s_z in (29).

It is interesting that even for $s = \frac{1}{2}$, we have not only one type of interaction, which respects dynamical symmetry, but two. Indeed, according to present consideration the Hamiltonian

$$\mathcal{H} = \frac{\mathbf{p^2}}{2m} + a\frac{s_x y - s_y x}{r^2} + b\frac{s_x x + s_y y}{r^2} \quad (30)$$

also possesses dynamical symmetry. The additional term in this Hamiltonian describes electric dipole in electric field $\frac{\vec{r}}{r^2}$ which is produced by rectilinear charge.

Last issue which we are going to discuss is the analogue of the formula (9) in the generic case. Defining as in (6) the operators J_i, having in mind discrete spectrum we obtain

$$\mathbf{J^2} + \frac{1}{4} = -\frac{m}{2}\frac{\mu(\mathbf{s},\mathbf{n})^2}{\mathcal{H}}. \quad (31)$$

Matrix $\mu(\mathbf{s},\mathbf{n})$ commutes with Hamiltonian, as it follows from it definition and the form of $\mathcal{H}$ (11) and from (31) we obtain

$$\mathcal{H} = -\frac{m}{2}\frac{\mu(\mathbf{s},\mathbf{n})^2}{\mathbf{J^2} + \frac{1}{4}}. \quad (32)$$

As matrix $\mu(\mathbf{s},\mathbf{n})$ in the bases $|s,k>$ is anti-diagonal, it square is diagonal

$$(\mu(\mathbf{s},\mathbf{n})^2)_{kk'} = diag\{|\alpha_s|^2, |\alpha_{s-1}|^2, ..., |\alpha_s|^2\}, \tag{33}$$

so it could be written as a linear combination of projectors on the states with definite values of s_z.

In conclusion we summarize the above discussion. It is shown that the problem, introduced in [1] for spin 1/2 admits generalization for arbitrary spin which preserves dynamical symmetry. The interaction depends on $2s+1$ parameters which leaves a big freedom for applications. The energy spectrum has the same character $1/n^2$ as in the case of spin1/2 , but for the wave functions we need to select the proper representations of $SO(3)$ corresponding to given value of spin. In [1] we have constructed explicitly invariant form of Schrodinger equation for our system as it was done by Fock [8] for Kepler problem. This form exists also for arbitrary spin. In the present paper we did not touche the subject of possible applications of the system we considered, as this is the theme for separate paper.

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